On the Number of Completions of Tumba

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Abstract. We discuss the number of ways to complete the game Tumba. We will prove that there are at least 10^{42} ways to complete the game. For comparison, there are approximately 10^{19} permutations of the Rubik cube, and 10^{26} metres from one end of the universe to the other. Empirical estimates show that the number of ways to complete Tumba is actually much higher, and could be equal to the number of particles in the universe.

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1. Introduction

This article discusses the number of ways to complete the game Tumba, invented by Noel Donegan and Luz Java [**1**], [**2**]. The game consists of 60 blocks, each with 5 colours, red, yellow, green, blue, and orange. We shall use R, Y, G, B, O to denote the colours. There are $5! = 120$ arrangements (or orderings, or permutations) of the 5 colours, however there are only 60 blocks since a block can be turned around to give the reverse ordering.

The game is to build a structure by placing blocks on top of previously placed blocks. Players take turns in drawing blocks blindly from a bag. The rule is that where two blocks touch, they must touch at squares of the same colour. Also, the structure must not fall down. A picture of a completed game is in Fig. 3.

The question we wish to consider is this:

What is the number of ways to complete the game Tumba?

It seems to be very difficult to give the *exact* answer to this question. The biggest obstacle to solving this problem is gravity. It is very hard to tell mathematically whether a given structure will fall down or not. Furthermore, another difficult problem we encounter is trying to

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determine (with a proof) whether a partially completed Tumba game can be fully completed using all 60 blocks. This is highly dependent on which blocks have been already chosen.

We shall give some lower bounds on the number of completions. These will prove that the number of completions is at least 10^{42}, but probably greater. Even if every person on earth played Tumba a million times each, the odds are overwhelmingly small that any two of those completions would be the same. *It is very safe to say that no two games of Tumba will ever be the same.* The odds of winning the lotto are much greater than the odds of two games of Tumba being the same. For another comparison, the observable universe is estimated to be 10^{26} metres from one end to the other. Thus, the number of ways to play Tumba is truly an astronomical number. The number of arrangements of the Rubik Cube is of 10^{19} order of magnitude, see [**3**], which is tiny in comparison.

We remind the reader of the notation

$$n! = n(n-1)(n-2)(n-3)\cdots(4)(3)(2)(1).$$

So $2! = 2$, $3! = 6$, $4! = 24$, and so on.

2. Completions in Zero Gravity

In zero gravity, the number of completions is highest, because we can ignore the effect of gravity and do not need to discount completions that would fall down. We can alternate placing blocks horizontally and vertically, as shown in Fig. 1.

Every block can be placed, so there are 60! ways to place the blocks.

Further, each horizontal block can be turned around, giving another 2^{30} arrangements.

This shows that the number of completions is at least

$$60! \times 2^{30} = 8934591879915433893810901742120573469607615789965\backslash$$
$$58231895504701225362155110400000000000000$$

which is approximately 8.9×10^{90}.

To get an idea of how big this number is, it is more than a billion times greater than the number of particles in the known universe, which is estimated to be 10^{80}.

3. One Particular Symmetric Completion

In this section we will describe a completion which is possible in theory. However in practice it may be impossible because it would fall down in everyday conditions. In perfect conditions, with a perfectly

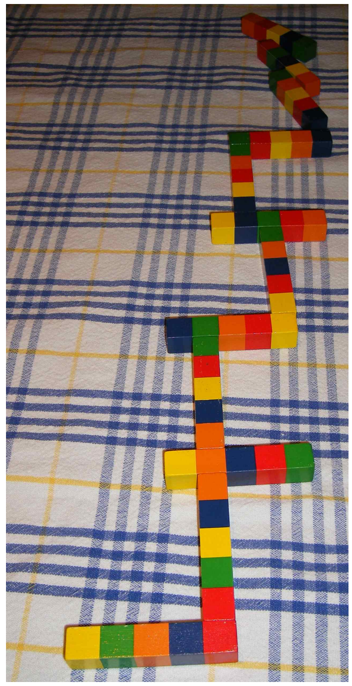


FIGURE 1. Section 2 Partial Construction in Zero Gravity

constructed set of blocks, this construction would not fall down. The reader may try this for themselves!

Consider all 12 blocks with R in the centre square. Let S_R denote the set of these 12 blocks, and similarly let S_Y, S_O, S_G, S_B be the same set for the other colours. Each of these 5 sets contains 12 blocks.

Begin with S_R, and place all of these horizontally on top of each other touching at the centre square. There are 12! different ways to do this.

Next choose a block in S_Y with R at one end, B at the other. There are 2 such blocks. Put this block vertically on top of the R centre square on the block at top of the build so far. The construction so far is shown in Fig. 2.

The top end of this block is B. Place all the elements of S_B horizontally on top of this B. There are another 12! ways to do this.

Next, choose an element of S_Y with B at one end, and G at the other end. Put this block vertically on top of the B centre square on the block at top of the build so far.

The top end of this block is G. Place all the elements of S_G horizontally on top of this G, except remove one block with Y at one end and O at the other end. We will need this block later.

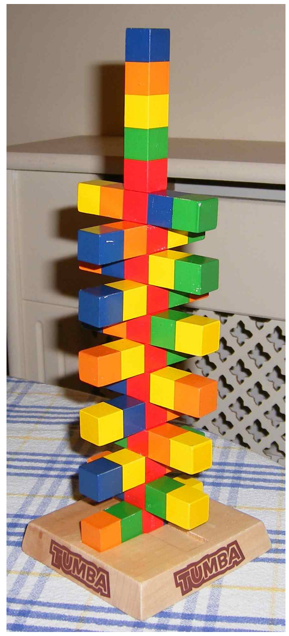


Figure 2. Section 3 Partial Construction

Next choose a block in S_Y with G at one end, O at the other. Put this block vertically on top of the G centre square on the block at top of the build so far.

The top end of this block is O. Place all the elements of S_O horizontally on top of this O.

Now place the special block we removed from S_G vertically on top of the centre O square. This special block has G in the middle and Y at the other end. Finally, place all the remaining 9 elements of S_Y horizontally on top of the Y end square.

The number of different ways of performing this particular construction is at least

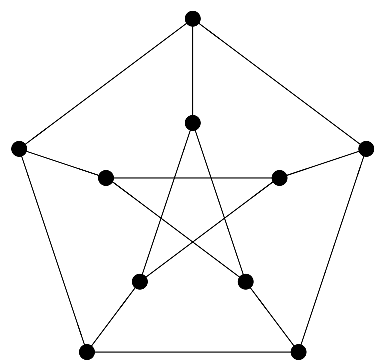
$$12! \times 12! \times 11! \times 12! \times 9! \times 2^4 \times 5! = 3056545547465739965614431016058880000000000$$

which is approximately 3×10^{42}.

So there are at least 10^{42} ways of completing Tumba. If everyone on earth played Tumba a billion times each, the odds are overwhelmingly small that any two of those completions would be the same. For another comparison, the observable universe is estimated to be 10^{26} metres across.

4. A Mathematical Completion

In this section we will describe one possible completion. The number of ways of carrying out this completion is related to the number of Hamiltonian paths in the Petersen graph, the graph shown below.



This section will be completed in the next version of this article. However, the number of completions here is much smaller than 10^{42}, so this section has mathematical interest but will not increase the number of completions beyond what we already know.

5. Highly Symmetric Completions versus Non-Symmetric

A completion with a high degree of symmetry has many different permutations of the blocks that give different completions. The exact number of such completions can be counted mathematically, as we have done earlier. Less symmetric completions, even including all their permutations, are more difficult to count. If these are more plentiful, however, then their number is important when estimating the order of magnitude of the total number of completions.

Any game of Tumba played blind will result in a random ordering of the 60 blocks, the order in which they are drawn from the bag. We need to know whether a high proportion of the 60! ways to choose blocks at random are completable. The number 60! has 81 digits, so is approximately 10^{81}. If 1 in 10 of these could be completed, then the number of completions would be approximately 10^{80}. It would be interesting to estimate the chances that a random ordering of the blocks can be built. This could be done by writing a computer programme to simulate the game, and running this a high number of times. This would make an interesting student project.

FIGURE 3. Completed Game

The best empirical estimate is obtained from the experience of the inventors. They have informed us that, in their experience of playing many games, there is at least a 70% chance that a randomly chosen ordering of the blocks can be built as a completed game. Therefore, 1 in 10 may be an underestimate, but assuming this figure we may conclude that Tumba has approximately 10^{80} completions, at least. This is approximately the number of particles in the universe. Needless to say, this is a phenomenally large number, which is hard to comprehend.

Taking into account different ways of playing each block, it is possible that the number of completions could reach 10^{100}, a number known as a googol.

6. A Real-Life Good Will Hunting

This is a true story. One evening I left my office with the blocks of Tumba scattered on the table. I locked my office door as usual. The following morning I unlocked the door, and entered my office to find the blocks had been built in a structure. I was so surprised I took a photograph, see Fig. 4. This was a way of building the blocks that I had not thought of. The only possible explanation is that the cleaner built the blocks when coming in to empty the rubbish bin!

The cleaner had not known the rules, and did not put colour on colour. Also, this method does not seem to work in the actual game,

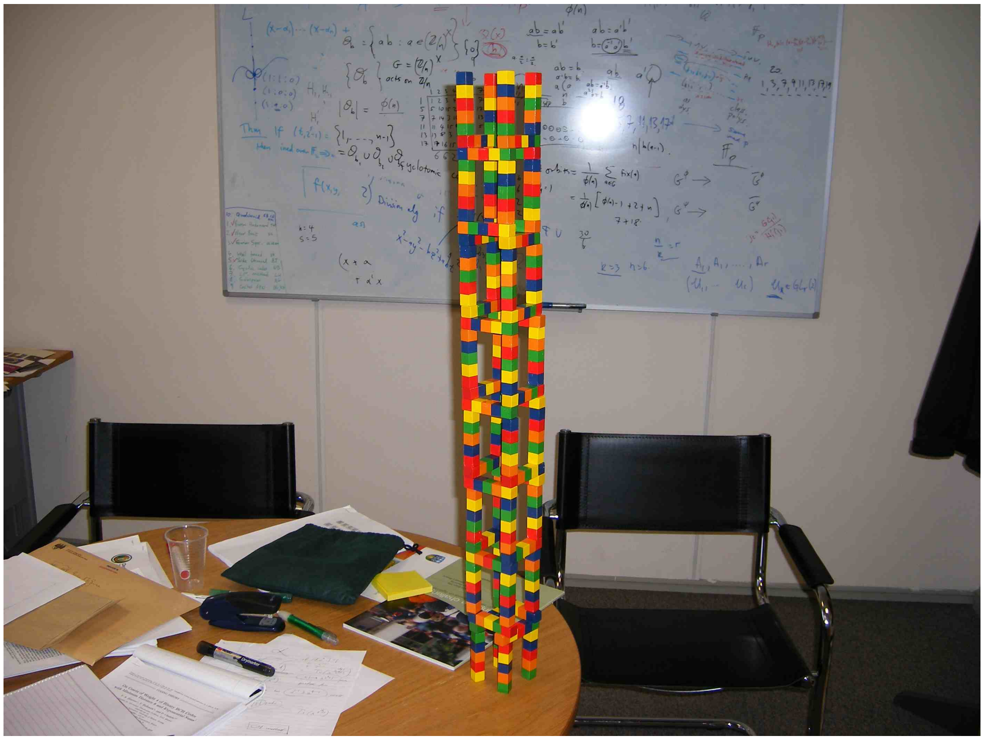


FIGURE 4. What the Janitor Did

because the game must start with one block. It does not seem possible to build this structure without a more stable starting point. Nevertheless, the idea is a good one and has given rise to other ideas about Tumba.

Acknowledgement We thank Carl Bracken for many helpful discussions about Tumba.

References

[1] www.luzjava.com
[2] http://www.rte.ie/tv/latelate/20061208.html
[3] The Rubik Cube, http://en.wikipedia.org/wiki/Rubik'sCube
[4] The Petersen Graph, book by D. A. Holton and J. Sheehan, Cambridge University Press.

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